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On the Automorphism Group of the Group $D_4(2)$

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1. INTRODUCTION

The present paper is concerned with the study of $\mathrm{Aut}(D_4(2))$, the automorphism group of the group $D_4(2)$ of the form $D_4(2)\,S_3$, where $D_4(2)$ is the simple Lie group of type D_4 and S_3 is the symmetric group on three symbols. It is well known that in the group $\bar{F}_{22}$, the automorphism group of the Fischer group F_{22}, there is an involution e such that its centralizer in F_{22} is isomorphic to the group $D_4(2)\,S_3$. In this paper we regard $D_4(2)\,S_3$ as a subgroup of F_{22} and we will calculate its conjugacy classes and character table.

2. PRELIMINARIES

Most of the notation is as in Moori [5] and Dye [1, 2]. Papers by Hunt [4], Steinberg [6] and Dye [1, 2] provide sufficient background material for this paper.

In [1], Dye calculated the character table of the simple group $FH(8,2)$ isomorphic to the special orthogonal group $O_8^+(2)$. The group $O_8^+(2)$ is isomorphic to the simple Lie group $D_4(2)$ of type D_4. The automorphism group of $D_4(2)$ is the group $D_4(2)\,S_3$ (Steinberg [6]) where $S_3 = \langle x, y\rangle$, and x and y are graph automorphisms of D_4 of orders 3 and 2, respectively.

In [2], Dye simultaneously with Frame [3] calculated the character table of the group $A = D_4(2)\langle y\rangle$, the full orthogonal group. Let us denote the group $D_4(2)\langle x\rangle$ by B. Then A and B are subgroups of $\mathrm{Aut}(D_4(2))$ and by using these subgroups we determined the conjugacy classes and the character table of $\mathrm{Aut}(D_4(2))$.

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3. Conjugacy Classes of $\mathrm{Aut}(D_4(2))$

3.1. *Sets of* $\mathrm{Aut}(D_4(2))$

The classes of $\mathrm{Aut}(D_4(2))$ split into three sets, which we denote by D-set, B-set and A-set and are defined in the following sections.

3.1.1. *The D-set and A-set*

The conjugacy classes of $D_4(2)$ fall into two sets (see Dye [1, table 6]); let us denote these sets by D_1 and D_2. Then D_1 consists of 14 classes of $D_4(2)$ which are fixed by S_3, and D_2 consists of 13×3 classes of $D_4(2)$ which in $\mathrm{Aut}(D_4(2))$ form a set D_2' of 13 classes.

Let $D = D_1 \cup D_2'$, then $|D| = 27$ and we have the following relations

TABLE I

Classes in the D-set

Type	Order	Order of centralizer	$\to F_{22}$
I	1	$2^{13} \cdot 3^6 \cdot 5^2 \cdot 7$	1
VIII	2	$2^{11} \cdot 3^2$	n
X	3	$2^4 \cdot 3^5$	ρ
XIII	2	$2^{13} \cdot 3^4$	ι
XV	4	$2^{10} \cdot 3^2$	4_4
XXVIII	4	$2^7 \cdot 3$	4_3
XXXIII	6	$3^4 \cdot 3^2$	$n\rho$
XXXIV	7	$2 \cdot 3 \cdot 7$	7
XXXV	8	$2^6 \cdot 3$	8_3
XXXVI	4	$2^{10} \cdot 3$	4_4
XXXVII	8	$2^6 \cdot 3$	8_2
XXXVIII	3	$2^4 \cdot 3^6$	ξ
XL	6	$2^4 \cdot 3^4$	$6_9 = \iota\xi$
XLII	12	$2^3 \cdot 3^3$	$12_7 = 4_4\xi$
III	2	$2^{11} \cdot 3^2 \cdot 5$	ι
IV	3	$2^7 \cdot 3^5 \cdot 5$	α
IX	6	$2^6 \cdot 3^2$	$6_6 = \iota\alpha$
XI	4	$2^7 \cdot 3$	4_2
XII	5	$2^3 \cdot 3 \cdot 5^2$	$5 = a$
XIV	6	$2^7 \cdot 3^3$	$6_6 = f\alpha$
XXVI	6	$2^4 \cdot 3^2$	$6_2 = \iota\rho$
XXVII	12	$2^4 \cdot 3$	$12_5 = 4_2 \cdot \alpha$
XXIX	10	$2^2 \cdot 5$	10_2
XXX	15	$2 \cdot 3 \cdot 5$	$15 = a \cdot \alpha$
XXXII	12	$2^5 \cdot 3^2$	$12_4 = 4_4 \cdot \alpha$
XXXIX	6	$2^4 \cdot 3^3$	$6_2 = \iota\rho$
XL_1	9	$2 \cdot 3^3$	9_1

between the orders of centralizers of the elements of D-set in the groups B, $D_4(2)$ and $\mathrm{Aut}(D_4(2))$:

(i) For $g \in D_1$, $|C_{\mathrm{Aut}(D_4(2))}(g) = 2\,|C_B(g)| = 6\,|C_{D_4(2)}(g)|$.

(ii) For $g \in D_2'$, $|C_{\mathrm{Aut}(D_4(2))}(g)| = 2\,|C_B(g)| = 2\,|C_{D_4(2)}(g)|$.

Conjugacy classes of the group A which are outside $D_4(2)$ form the A-set for the classes of $\mathrm{Aut}(D_4(2))$. There are 27 such classes (see Dye [2]) and for any g in this set we have

$$C_{\mathrm{Aut}(D_4(2))}(g) = C_A(g).$$

3.1.3. *The B-Set*

Conjugacy classes of the group B outside $D_4(2)$ are of types dx and dx^2 where $d \in D_4(2)$. Under the action of S_3 each pair of dx and dx^2 are fused

TABLE II

Classes in the A-Set

Type	Order	Order of centralizer	$\to F_{22}$
II	2	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	1
V	2	$2^{10} \cdot 3^2$	n
VI	6	$2^5 \cdot 3^3 \cdot 5$	$d\alpha$
VII	4	$2^9 \cdot 3 \cdot 5$	4_1
XVI	6	$2^5 \cdot 3^2$	$6_7 = n\alpha$
XVII	6	$2^3 \cdot 3^3$	$6_1 = d\rho$
XVIII	4	2^8	4_2
XIX	12	$2^5 \cdot 3$	$12_3 = 4_1 \cdot \alpha$
XX	10	$2^2 \cdot 3 \cdot 5$	$10_1 = ad$
XXI	4	$2^8 \cdot 3$	4_5
XXII	6	$2^3 \cdot 3^2$	$6_9 = n\rho$
XXIII	4	$2^9 \cdot 3^2$	4_1
XXIV	12	$2^5 \cdot 3^2$	$4_1\alpha = 12_3$
XXV	8	$2^6 \cdot 3$	8_2
XLIII	30	$2 \cdot 3 \cdot 5$	$30 = a\,d\alpha$
XLIV	20	$2^2 \cdot 5$	$20 = 4_1 \cdot a$
XLV	14	$2 \cdot 7$	$14 = 7d$
XLVI	12	$2^3 \cdot 3^2$	$12_1 = 4_1\rho$
XLVII	24	$2^3 \cdot 3$	$24_2 = 8_2 \cdot \alpha$
XLVIII	8	2^4	8_1
XLIX	6	$2^4 \cdot 3^4$	$6_8 = d\xi$
L	6	$2^4 \cdot 3^2$	$6_{10} = n\xi$
LI	18	$2 \cdot 3^2$	$18_1 = d9_1$
LII	12	$2^3 \cdot 3$	$12_8 = 4_5\xi$
LIII	4	$2^8 \cdot 3$	4_1
LIV	12	$2^3 \cdot 3$	$12_1 = 4_1\rho$
LV	8	2^6	8_3

together and they form a single class in $\mathrm{Aut}(D_4(2))$. We will denote the set of these classes in $\mathrm{Aut}(D_4(2))$ by B-set and for any g in this set we have

$$C_{\mathrm{Aut}(D_4(2))}(g) = C_B(g).$$

It is easy to see that the number of conjugacy classes of the group B is $3 \times 14 + 13 = 55$. Since $14 + 13$ of these classes are inside $D_4(2)$, the number of classes of B outside $D_4(2)$ is 28 and hence the number of classes of $\mathrm{Aut}(D_4(2))$ in the B-set is 14. Therefore for the calculation of the conjugacy classes of $\mathrm{Aut}(D_4(2))$ we need only determine these 14 classes of the B-set.

The group $\mathrm{Aut}(D_4(2))$ is a subgroup of F_{22} of index 61776. By using the character tables of the groups F_{22} (Hunt [4]), A and $D_4(2)$ (Dye [1, 2]) we have found the correspondence between the classes of D-set, A-set and of F_{22}. Using the same notation employed by Dye in [1, 2], we have listed these classes in the first columns of Tables I and II. Their identification among the classes of F_{22} is given in the last columns of these tables.

Let χ be the permutation character of F_{22} on $\mathrm{Aut}(D_4(2))$, then we have

$$\chi = \underline{1} + \underline{3080} + \underline{13650} + \underline{45045},$$

where $\underline{m}$ denotes the irreducible character of degree m in F_{22}. Now by using the values of χ and the correspondence between the classes of F_{22} and classes in the D-set and A-set we have determined the classes in the B-set. The complete list of these classes together with the power maps and identification among classes of F_{22} are given in Table III. The third powers of these classes are contained in the D-set and are listed in the last column of this table.

TABLE III

Classes in the B-Set

β	3	$2^3 \cdot 3^4$	β	—	—
$6_{(\beta)}$	6	$2^3 \cdot 3^2$	$6_{(11)}$	β	VIII
$12_{(\beta)}$	12	$2^2 \cdot 3$	$12_{(11)}$	$6_{(\beta)}$	XXVIII
$9_{(2)}$	9	$2 \cdot 3^3$	$9_{(2)}$	—	XXXVIII
$18_{(2)}$	18	$2 \cdot 3^2$	$18_{(4)}$	$9_{(2)}$	XL
α'	3	$2^6 \cdot 3^4 \cdot 7$	α	—	—
$6_{(\alpha')1}$	6	$2^6 \cdot 3^2$	$6_{(6)}$	α'	XIII
$6_{(\alpha')2}$	6	$2^4 \cdot 3^2$	$6_{(7)}$	α'	VIII
$12_{(\alpha')1}$	12	$2^5 \cdot 3$	$12_{(4)}$	$6_{(\alpha')1}$	XXXVI
$12_{(\alpha')2}$	12	$2^5 \cdot 3^2$	$12_{(4)}$	$6_{(\alpha')1}$	XV
$12_{(\alpha')3}$	12	$2^4 \cdot 3^2$	$12_{(6)}$	$6_{(\alpha')1}$	XV
$24_{(\alpha')1}$	24	$2^3 \cdot 3$	$24_{(1)}$	$12_{(\alpha')3}$	XXXV
$24_{(\alpha')2}$	24	$2^3 \cdot 3$	$24_{(2)}$	$12_{(\alpha')1}$	XXXVII
21	21	$3 \cdot 7$	21	—	XXXIV

4. The Character Table of $\mathrm{Aut}(D_4(2))$

4.1. *The Structure of the Character Table*

In the group $D_4(2)$ we have 14 self-trial irreducible characters χ_i, $1 \leqslant i \leqslant 14$ and 13 trial irreducible characters $\{\chi^{(a)}, \chi^{(b)}, \chi^{(c)}\}$ (see Dye [1]). For each χ_i, $1 \leqslant i \leqslant 14$, in the group B we will have three irreducible characters χ_i, $\chi_i^{(\alpha)}$ and $\chi_i^{(\bar{\alpha})}$ where $\alpha = \frac{1}{2}(-1 + \sqrt{3})$

$$\chi_i(g) = \chi_i^{(\alpha)}(g) = \chi_i^{(\bar{\alpha})}(g) \qquad \text{for} \quad g \in D_4(2)$$

and

$$\chi_i^{(\alpha)}(g) = \chi_i(g)\alpha, \qquad \chi_i^{(\bar{\alpha})}(g) = \chi_i(g)\bar{\alpha} \qquad \text{for} \quad g \in B \backslash D_4(2).$$

Each χ_i $1 \leqslant i \leqslant 14$, of the group B, will produce a pair of associate irreducible characters χ_i and χ_i' for the group $\mathrm{Aut}(D_4(2))$ where

$$\chi_i(g) = \chi_i'(g) \qquad \text{for} \quad g \in (D\text{-set}) \cup (B\text{-set})$$

and

$$\chi_i(g) = -\chi_i'(g) \qquad \text{for} \quad g \in A\text{-set}.$$

For each pair of irreducible characters $\chi_i^{(\alpha)}$, $\chi_i^{(\bar{\alpha})}$ of the group B, we will have an irreducible character $(\chi_i^{(\alpha)} + \chi_i^{(\bar{\alpha})})$ for the group $\mathrm{Aut}(D_4(2))$ such that

$$\begin{aligned} (\chi_i^{(\alpha)} + \chi_i^{(\bar{\alpha})})(g) &= 2\chi_i(g) && \text{if} \quad g \in D\text{-set} \\ &= -\chi_i(g) && \text{if} \quad g \in B\text{-set} \\ &= 0 && \text{if} \quad g \in A\text{-set}. \end{aligned}$$

The trial irreducible characters $\{\chi^{(a)}, \chi^{(b)}, \chi^{(c)}\}$ of $D_4(2)$ fuse together in the group B and give rise to an irreducible character $\chi^{(a,b,c)}$ where

$$\begin{aligned} \chi^{(a,b,c)}(g) &= (\chi^{(a)} + \chi^{(b)} + \chi^{(c)})(g) && \text{if} \quad g \in D_4(2) \\ &= 0 && \text{otherwise}. \end{aligned}$$

We have 13 irreducible characters of type $\chi^{(a,b,c)}$ in the group B and each of these give rise to a pair of associate characters $\chi^{(a,b,c)}$ and $\chi^{(a,b,c)\prime}$ in the $\mathrm{Aut}(D_4(2))$ where

$$\begin{aligned} \chi^{(a,b,c)}(g) &= (\chi^{(a)} + \chi^{(b)} + \chi^{(c)})(g) && \text{if} \quad g \in D\text{-set} \\ &= 0 && \text{if} \quad g \in B\text{-set} \\ &= \chi^{(a)}(g) && \text{if} \quad g \in A\text{-set} \end{aligned}$$

and

$$\begin{aligned}\chi^{(a,b,c)\prime}(g) &= \chi^{(a,b,c)}(g) \qquad \text{if} \quad g \in (D\text{-set}) \cup (B\text{-set}) \\ &= -\chi^{(a)}(g) \qquad \text{if} \quad g \in A\text{-set}.\end{aligned}$$

Now by using the above results we are able to partition the character table of Aut($D_4(2)$) into the following 15 "blocks":

	D-set	B-set	A-set
14 characters of type x_i	$\Delta(D)$	$\Delta(B)$	$\Delta(A)$
14 characters of type χ_i'	$\Delta(D)$	$\Delta(B)$	$-\Delta(A)$
14 characters of type $(\chi_i^{(a)} + \chi_i^{(\bar{a})})$	$2\Delta(D)$	$-\Delta(B)$	0
13 characters of type $\chi^{(a,b,c)}$	$\Delta(a, b, c)$	0	$\Delta(a)$
13 characters of type $\chi^{(a,b,c)\prime}$	$\Delta(a, b, c)$	0	$-\Delta(a)$

Since the character tables of $D_4(2)$ and A are known, the "blocks" $\Delta(D)$, $\Delta(a, b, c)$, $\Delta(a)$ and $\Delta(A)$ are known. Hence to calculate the character table of Aut($D_4(2)$) we need only determine $\Delta(B)$; this means we have to calculate the values of χ_i, $1 \leqslant i \leqslant 14$ on all classes in the B-set.

4.2. *Orthogonality Relations on the B-set*

By using the structure of the character table of Aut($D_4(2)$) described in (4.1), we have obtained the following orthogonality relations for characters of Aut($D_4(2)$) on the B-set. For each χ_i, $1 \leqslant i \leqslant 14$ we have

$$\frac{1}{|\mathrm{Aut}(D_4(2))|} \sum_{g \in B\text{-set}} \chi_i(g)\chi_i(g^{-1}) = \frac{2}{6}. \tag{4.2.1}$$

For χ_i, χ_j, $1 \leqslant i \leqslant 14$, $1 \leqslant j \leqslant 14$ and $i \neq j$ we have

$$\sum_{g \in B\text{-set}} \chi_i(g)\chi_j(g^{-1}) = 0. \tag{4.2.2}$$

Now suppose ψ is a character of Aut($D_4(2)$) and we have the following decomposition for ψ in terms of irreducible characters of Aut($D_4(2)$):

$$\begin{aligned}\psi = \sum_{i=1}^{14} a_i\chi_i + \sum_{i=1}^{14} a_i'\chi_i' + \sum_{i=1}^{14} b_i(\chi_i^{(a)} + \chi_i^{(\bar{a})}) \\ + \sum_{k=1}^{13} c_k\chi_k^{(a,b,c)} + \sum_{k=1}^{13} c_k'\chi_k^{(a,b,c)\prime}.\end{aligned}$$

Then it is easy to see that

$$\psi(g) = \sum_{i=1}^{14} (a_i + a_i' - b_i)\,\chi_i(g) \qquad \text{for} \quad g \in B\text{-set}, \tag{4.2.3}$$

$$(\psi, \psi)_{\mathrm{Aut}(D_4(2))} = (\psi, \psi)_A + (\psi, \psi)_{B\text{-set}} - \tfrac{2}{6}(\psi, \psi)_{D_4(2)} \tag{4.2.4}$$

and

$$(\psi, \psi)_{B\text{-set}} = \frac{2}{6} \sum_{i=1}^{14} (a_i + a_i' - b_i)^2. \tag{4.2.5}$$

4.3. *The Restriction of the Irreducible Characters of* F_{22}

Let ψ be an irreducible character of F_{22}. We can determine $\psi|_{\mathrm{Aut}(D_4(2))}$ in terms of irreducible characters of $\mathrm{Aut}(D_4(2))$ by the following steps.

(i) Since $D_4(2) \leqslant F_{22}$, by using the embedding of $D_4(2)$ in F_{22} and the character table of $D_4(2)$ we can evaluate $\psi|_{D_4(2)}$ in terms of irreducible characters of $D_4(2)$.

(ii) By using the embedding of B-set in F_{22}, we are able to calculate the values of ψ on B-set and hence we can determine the $(\psi, \psi)_{B\text{-set}}$.

(iii) By using (i), (ii), (4.2.5) and the values of ψ on A-set we can determine $\psi|_{\mathrm{Aut}(D_4(2))}$ in terms of irreducible characters of $\mathrm{Aut}(D_4(2))$.

Hence for each irreducible character ψ of F_{22}, we will have a relation of type (4.2.3) on the B-set. For example, let $\psi = \underline{429}$ be the irreducible character of degree 429 in F_{22} (Hunt [4]). Then we have

$$\psi|_{D_4(2)} = 2\chi_1 + \chi_3 + \chi^{(2)} + \chi^{(3)} + \chi^{(4)},$$

where χ_1 is the identity character and $\chi_3 = \chi^{(8)}$ according to the Dye's notation in Dye [1], and

$$(\psi, \psi)_{B\text{-set}} = \tfrac{4}{6}.$$

Hence by (4.2.5), we have

$$(a_1 + a_1' - b_1)^2 + (a_3 + a_3' - b_3)^2 = 2$$

which implies

$$a_1 + a_1' - b_1 = a_3 + a_3' - b_3 = 1.$$

Now using the values of ψ on the A-set, we have

$$a_1 = a_1' = a_3' = b_3 = c_k' = 0, \qquad a_3 = b_1 = 1.$$

TABLE IV

$\Delta(B)$

B-set / Characters	β	$6_{(\beta)}$	$12_{(\beta)}$	$9_{(2)}$	$18_{(2)}$	α'	$6_{(\alpha')1}$	$6_{(\alpha')2}$	$12_{(\alpha')1}$	$12_{(\alpha')2}$	$12_{(\alpha')3}$	$24_{(\alpha')1}$	$24_{(\alpha')2}$	21
$\chi_1 = \chi^{(0)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_2 = \chi^{(1)}$	−1	−1	−1	2	0	8	0	2	0	4	−2	0	0	1
$\chi_3 = \chi^{(8)}$	−2	2	0	1	1	7	7	−1	−1	−1	−1	−1	−1	0
$\chi_4 = \chi^{(15)}$	−2	2	0	1	−1	7	−1	−1	3	−1	−1	1	−1	0
$\chi_5 = \chi^{(19)}$	3	−1	1	0	0	21	−3	−1	1	5	−1	−1	−1	0
$\chi_6 = \chi^{(21)}$	−2	2	0	1	−1	7	−1	−1	−1	3	3	−1	1	0
$\chi_7 = \chi^{(27)}$	0	0	0	0	0	27	3	3	−1	3	3	1	−1	−1
$\chi_8 = \chi^{(28)}$	0	0	0	0	0	27	3	3	3	−1	−1	−1	1	−1
$\chi_9 = \chi^{(29)}$	5	1	−1	−1	1	14	−2	−2	2	2	2	0	0	0
$\chi_{10} = \chi^{(33)}$	2	−2	0	2	0	56	0	−2	0	−4	2	0	0	0
$\chi_{11} = \chi^{(43)}$	6	2	0	0	0	42	−6	2	−2	−2	−2	0	0	0
$\chi_{12} = \chi^{(44)}$	−8	0	0	−2	0	64	0	0	0	0	0	0	0	1
$\chi_{13} = \chi^{(51)}$	8	0	0	−1	−1	8	8	0	0	0	0	0	0	1
$\chi_{14} = \chi^{(52)}$	0	0	0	0	0	27	3	−3	−1	3	−3	1	1	−1

Therefore

$$\psi|_{\mathrm{Aut}(D_4(2))} = (\chi_1^{(\alpha)} + \chi_1^{(\bar{\alpha})}) + \chi_3 + \chi^{(2,3,4)}$$

and by (4.2.3) we have

$$\psi(g) = (-\chi_1 + \chi_3)(g) \qquad \text{for} \quad g \in B\text{-set}.$$

4.4. *The Values of* χ_{12}

Since χ_{12} is of degree $4096 = 2^{12}$, the character $(\chi_{12}^{(\alpha)} + \chi_{12}^{(\bar{\alpha})})$ has 2^{13} as its degree. The order of a Sylow 2-subgroup of $\mathrm{Aut}(D_4(2))$ is equal to 2^{13}, so the character $(\chi_{12}^{(\alpha)} + \chi_{12}^{(\bar{\alpha})})$ has values zero on all the 2-singular classes of $\mathrm{Aut}(D_4(2))$. Now by (4.1) we have

$$(\chi_{12}^{(\alpha)} + \chi_{12}^{(\bar{\alpha})})(g) = -\chi_{12}(g) \qquad \text{for} \quad g \in B\text{-set}$$

and hence χ_{12} has values zero on all 2-singular classes of B-set. The only 2-regular classes in the B-set are β, $9_{(2)}$, α', 21 (see Table III); and by using the structure of 7-blocks of defect one it is easy to see that $\chi_{12}(21) = 1$.

Now by using the relations (4.2.1) and (4.2.2) for χ_{12} we will have

$$\chi_{12}(\beta) = -8, \qquad \chi_{12}(9_{(2)}) = -2 \qquad \text{and} \qquad \chi_{12}(\alpha') = 64.$$

4.5. *The Values of* χ_i, $1 \leqslant i \leqslant 14$, $i \neq 12$.

We have restricted the characters of F_{22} to $\mathrm{Aut}(D_4(2))$, and studied the relations of type (4.2.3) obtained from these characters by the method which we described in (4.3).

By using these relations and the values of χ_{12} together with the symmetric and alternative powers of irreducible characters χ_2 and χ_4, we were able to calculate the values of χ_i, $1 \leqslant i \leqslant 14$ on the B-set.

The complete list of χ_i, $1 \leqslant i \leqslant 14$ and thei values on the B-set are given in Table IV. In the first column of this table, Dye's notation for these characters are also listed.

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